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DIFFERENTIAL ENCOUNTER-EVASION GAME FOR A PARABOLIC SYSTEM UNDER INTEGRAL CONSTRAINTS ON THE PLAYER'S CONTROLS

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The problem of controlling a parabolic system under conditions of uncertainty or conflict is analyzed. The problem is treated as a position differential game in a suitable functional space [1—3]. The controls occur in the boundary condition; the mechanism for developing these controls is described by an ordinary differential equation. The constructions are based on an approach to position control problems in distributed-parameter systems given in [3—5] (*). As in the case of ordinary differential equations [6] a procedure of control with a guide is constructed, yielding the solution of the problems being analyzed. The paper is closely related to the researches in [1—9].

1. Statement of the problem. Let Ω be a bounded connected open set in the Euclidean space R_n and Γ be the boundary of Ω. We assume that constraints (see [10], pp. 212 and 222) ensuring sufficient smoothness of the solutions of the boundary-value problems to be examined have been imposed on domain Ω. We consider the conflict-controlled system

$$\partial y/\partial t + Ay = f, \text{ in } Q = (t_0, \vartheta) \times \Omega \tag{1.1}$$

$$y|_{t=t_0} = y_0, \text{ in } \Omega \tag{1.2}$$

$$y|_\Sigma = \alpha(x)w(t), \text{ in } \Sigma = (t_0, \vartheta) \times \Gamma$$

$$dw/dt = B(t)w + C(t)u + D(t)v, \quad w(t_0) = w_0 \tag{1.3}$$

where A is a self-adjoint elliptic operator of the form

$$Ay = -\sum_{ij=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j} y\right) + a(x)y \tag{1.4}$$

Here $y_0 \in L_2(\Omega)$ is a specified initial state, $f \in L_2(Q)$ is a specified perturbation; $\alpha(x)$ is an m-dimensional vector function square-summable with respect to Γ; w is the m-dimensional phase vector of system (1.3); u and v are vector-valued controls of dimensions l_1 and l_2, respectively; $B(t)$, $C(t)$ and $D(t)$ are continuous matrices of appropriate dimensions. The functions $u(\cdot)$ $(v(\cdot))$, Lebesgue measurable on $[t_0, \vartheta]$, satisfying the following constraints ($\|g\|$ is the Euclidean norm of vector g):

$$J_1(u, t_0, \vartheta) = \left(\int_{t_0}^{\vartheta} \|u(t)\|^{p_1} dt\right)^{1/p_1} \leqslant \mu(t_0) \tag{1.5}$$

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*) Osipov, Iu. S., Differential games in distributed parameter systems. Abstracts Third All-Union Conf. Game Theory. Odessa, 1974.

$$\left(J_2(v, t_0, \vartheta) = \left(\int_{t_0}^{\vartheta} \|v(t)\|^{p_2} dt\right)^{1/p_2} \leqslant \nu(t_0)\right)$$

$$\mu(t_0) < \infty, \ \nu(t_0) < \infty; \ 1 < p_i < \infty, \ i = 1, 2$$

are called the admissible controls of the first (second) player. The variation of the constraints $\mu(t)$ and $\nu(t)$ is determined by resources of controls u and v expended during the game

$$\mu^{p_1}(t_2) = \mu^{p_1}(t_1) - J_1^{p_1}(u, t_1, t_2) \tag{1.6}$$

$$\nu^{p_2}(t_2) = \nu^{p_2}(t_1) - J_2^{p_2}(v, t_1, t_2), \quad t_2 > t_1$$

A closed set M is specified in space $L_2(\Omega)$. Find a method for selecting the control u (the control v) on the feedback principle, $u[t] = u[t, y_t(\cdot)]$ $(v[t] = v[t, y_t(\cdot)])$, developing the realizations $u[t]$ $(v[t])$, Lebesgue measurable on $[t_0, \vartheta]$, and satisfying (1.5), such that the condition $y_\vartheta(\cdot) \in M$ $(y_\vartheta(\cdot) \notin M)$ for any admissible control $v(\cdot)$ $(u(\cdot))$ is satisfied. Here $y_t(\cdot) = y(t, x)$, $x \in \Omega$ is the state of system (1.1) at instant t.

Position control problems in such a setting were considered in [1−5], wherein the case of instantaneous constraints on the controls was studied. The case of constraints (1.5) and (1.6) was studied also in [6] for controlled systems described by ordinary differential equations. In the present paper, as in [5], we study a parabollic controlled system in the presence of constraints (1.5) and (1.6). It is sensible to treat the problems to be examined as problems of the optimal heating of body Ω under conditions of uncertainty on the means of conducting heat to the body from some heat source distributed on boundary Γ and described by an ordinary differential equation. Problems of similar form arise, for example, when heating a metal under rolling or heat treatment (see [11]).

Let us make the problem statement more precise. By a solution of system (1.1) with selected y_0, w_0, $u(t)$ and $v(t)$ we mean the function $y_t(x) = y(t, x; y_0, w_0, u(\cdot), v(\cdot)) \in L_2(Q)$ satisfying the following integral identity [12]:

$$\int_{t_0}^{\vartheta}\int_{\Omega} y_t(x)\left(-\frac{\partial \varphi}{\partial t} + A\varphi\right) dx\, dt = \int_{t_0}^{\vartheta}\int_{\Omega} f\varphi\, dx\, dt + \tag{1.7}$$

$$\int_{\Omega} y_0(x)\,\varphi(x, t_0)\, dx - \int_{t_0}^{\vartheta}\int_{\Gamma} [\alpha(x)\, w(t)]\, \frac{\partial \varphi}{\partial \nu_A}\, d\Gamma\, dt$$

$$\forall \varphi \in X = \{\varphi \mid \varphi \in H^{2,1}(Q);\ \varphi|_\Sigma = 0,\ \varphi(x, \vartheta) = 0,\ x \in \Omega\}$$

Here $w(t)$ is a solution of the integral equation

$$w(t) = w_0 + \int_{t_0}^{t} B(\tau)\, w(\tau)\, d\tau + \int_{t_0}^{t} C(\tau)\, u(\tau)\, d\tau + \int_{t_0}^{t} D(\tau)\, v(\tau)\, d\tau \tag{1.8}$$

Set X is provided with a topology induced by the topology of space $H^{2,1}(Q)$ [12]

$$H^{2,1}(Q) = \{\varphi | \varphi,\ \partial\varphi / \partial t,\ \partial\varphi / \partial x_i,\ \partial^2\varphi / \partial x_i \partial x_j \in L_2(Q),\ i, j = 1, \ldots, n\}$$

The set of motions introduced is not empty [12]. Using the expansion of solution $y_t(x)$ with respect to the basis made up of the eigenfunctions of operator A, it can be shown that function $y_t(x)$ is continuous in t and has values in space $L_2(\Omega)$. The vector

$r = \{w, y, \mu, \nu\}$, where $w \in R_m$, $y \in L_2(\Omega)$, $\mu \geqslant 0$ and $\nu \geqslant 0$, is called the system's state. The pairs $\{t, r\}$, $t \in [t_0, \vartheta]$ are called positions. Let Δ be a covering of the interval $[t_0, \vartheta]$ by the semi-intervals $[\tau_i, \tau_{i+1})$ of equal length

$$\delta = \delta(\Delta) = \tau_{i+1} - \tau_i \quad (t_0 = \tau_0 < \tau_1 < \ldots < \tau_{m(\Delta)} = \vartheta)$$

By knowning the system's state at instant τ_i, i.e. $r(\tau_i) = \{w(\tau_i), y_{\tau_i}(\cdot), \mu(\tau_i), \nu(\tau_i)\}$, the first (second) player selects on the semi-interval $[\tau_i, \tau_{i+1})$ a control $u(\cdot)$ $(v(\cdot))$ admissible for the position $\{\tau_i, r(\tau_i)\}$, i.e.

$$J_1(u, \tau_i, \tau_{i+1}) \leqslant \mu(\tau_i), \quad J_2(v, \tau_i, \tau_{i+1}) \leqslant \nu(\tau_i)$$

Such a method of selecting u (v) is called a strategy U (V) of the first (second) player. The function $y_t(x)_\Delta = y(t, x; w_0, y_0, \mu(t_0), \nu(t_0), U)_\Delta$ satisfying (1.7) is called the motion of system (1.1) from the position $\{t_0, w_0, y_0, \mu(t_0), \nu(t_0)\}$, corresponding to the partitioning Δ of the interval $[t_0, \vartheta]$ and to strategy U. Here the control u is selected on the semi-interval $[\tau_i, \tau_{i+1})$ by strategy U with respect to the position $\{\tau_i, w(\tau_i), y_{\tau_i}(\cdot), \mu(\tau_i), \nu(\tau_i)\}$ and $v(\cdot)$ is an admissible realization of the second player's control. The motions $y(t, x; w_0, y_0, \mu(t_0), \nu(t_0), V)_\Delta$ are defined similarly. The problems set for the players are formalized as follows.

The encounter problem. Construct the first player's strategy U with the property: for any $\varepsilon > 0$ a positive number δ_0 can be found such that the inclusion $y_\vartheta(\cdot)_\Delta \in M^\varepsilon$ (M^ε is the closed ε-neighborhood of set M) is accomplished for all motions $y_t(x)_\Delta = y(t, x; w_0, y_0, \mu(t_0), \nu(t_0), U)_\Delta$ if only $\delta(\Delta) \leqslant \delta_0$.

The evasion problem. Construct the second player's strategy V with the property: for any $\varepsilon > 0$ a positive number δ_0 can be found such that the inclusion $y_\vartheta(\cdot)_\Delta \notin M^\varepsilon$ is accomplished for all motions $y_t(x)_\Delta = y(t, x; w_0, y_0, \mu(t_0), \nu(t_0), V)_\Delta$ if only $\delta(\Delta) \leqslant \delta_0$.

2. Let us describe the first player's control procedure yielding a solution to the encounter problem. This procedure is similar to the control with a guide procedure (see [2, 6]). By H we denote the space

$$H = R_m \times L_2(\Omega) \times R_1 \times R_1$$

with the norm

$$\|\{w, y, \mu, \nu\}\|_H = (\|w\|^2_{R_m} + \|y\|^2_{L_2(\Omega)} + \mu^2 + \nu^2)^{1/2}$$

Let $z_* = \{t_*, w_*, y_*, \mu_*, \nu_*\}$ be some position of the game and $v[t]$, $t_* \leqslant t \leqslant \vartheta$ be a realization of the second player's control, admissible for position z_*, i.e. $J_2(v, t_*, \vartheta) \leqslant \nu_*$. Following [6], by $G^{(u)}(z_*, t^*, v[\cdot])$, $t^* > t_*$ we denote the set of points $r(t^*) = \{w(t^*), y_{t^*}(\cdot), \mu(t^*), \nu(t^*)\}$, where $\mu(t^*) \geqslant 0$, $\mu^{p_1}(t^*) \leqslant \mu_*^{p_1} - J_1^{p_1}(u, t_*, t^*)$, $w(t)$ is a solution of Eq. (1.8) with $u(t)$ and $v[t]$; $y_{t^*}(x) = y(t^*, x; y_*, w_*, u(\cdot), v[\cdot])$, $\nu^{p_2}(t^*) = \nu_*^{p_2} - J_2^{p_2}(v, t_*, t^*)$. Here the $u(\cdot)$ are all possible summable functions satisfying the constraint $J_1(u, t_*, t^*) \leqslant \mu_*$. By M^* we denote the set

$$M^* = \{\{w, y, \mu, \nu\} \in H \mid y \in M, \ \mu \geqslant 0, \ \nu \geqslant 0\} \tag{2.1}$$

Let a family of sets N_t, $t_0 \leqslant t \leqslant \vartheta$, be given in space H. We say that the system of sets N_t is strongly u-stable if the condition

$$G^{(u)}(\{t_1, w_1, y_1, \mu_1, \nu_1\}, t_2, v(\cdot)) \cap N_{t_2} \neq \varnothing \tag{2.2}$$

is satisfied for any t_1 and t_2 $(t_0 \leqslant t_1 < t_2 \leqslant \vartheta)$, for any $\{w_1, y_1, \mu_1, \nu_1\} \in N_{t_1}$ and for every function $v(\cdot)$ such that $J_2(v, t_1, t_2) \leqslant \nu_1$. Let the function $u_*(z, z^*, \delta)$ minimize the integral

$$\int_{t_*}^{t_*+\delta} b'u(t)\,dt \tag{2.3}$$

for $J_1^{p_1}(u, t_*, t_*+\delta) \leqslant \mu^{p_1} - \mu^{*p_1}$, $\mu - \mu^* > 0$, $\|b\| \neq 0$ and $u_*(z, z^*, \delta) = 0$ when $\mu - \mu^* \leqslant 0$ or when $\|b\| = 0$. The function $v^*(z, z^*, \delta)$ maximizes the integral

$$\int_{t_*}^{t_*+\delta} c'v(t)\,dt \tag{2.4}$$

for $J_2^{p_2}(v, t_*, t_*+\delta) \leqslant \nu^{*p_2} - \nu^{p_2}$, $\nu^* - \nu > 0$, $\|c\| \neq 0$ and $v^*(z, z^*, \delta) = 0$ when $\nu^* - \nu \leqslant 0$ or when $\|c\| = 0$. Here

$$z = \{t_*, w, y, \mu, \nu\}, \quad z^* = \{t_*, w^*, y^*, \mu^*, \nu^*\}$$
$$\delta > 0, \ b = (w - w^*)'C(t_*), \quad c = (w - w^*)'D(t_*)$$

(the prime denotes transposition).

The following data are specified: the initial position $z(t_*) = \{t_*, w_*, y_*, \mu_*, \nu_*\}$ and a system of strongly u-stable sets N_t, $t_* \leqslant t \leqslant \vartheta$. We choose a position

$$z^*(t_*) = \{t_*, w^*, y^*, \mu^*, \nu^*\} \in \{t = t_*\} \times N_{t*}$$

arbitrarily. This is the position of an auxiliary motion, viz., of the guide (see [2, 6]) at instant $t = t_*$. We select a covering Δ of the interval $[t_*, \vartheta]$ by a system of semi-intervals $[\tau_i, \tau_{i+1})$ of equal length $\delta = \delta(\Delta) = \tau_{i+1} - \tau_i$ $(t_* = \tau_0 < \tau_1 < \ldots < \tau_{m(\Delta)} = \vartheta)$. We assume that on the first segment $[\tau_0, \tau_1)$ the motion of system (1.1)–(1.3) is generated by the first player's control

$$u^{(0)}[t] = u_*(z(t_*), z^*(t_*), \delta), \quad \tau_0 \leqslant t < \tau_1$$

in pair with a certain realization $v[t]$ of the second player's control, admissible for the position $z(t_*)$, i.e. $J_2(v, \tau_0, \tau_1) \leqslant \nu_*$. The choice of these controls determines the game's position

$$z(\tau_1) = \{\tau_1, w(\tau_1), y_{\tau_1}(\cdot), \mu(\tau_1), \nu(\tau_1)\}$$

reached at the instant $t = \tau_1$. We select the guide's position $z^*(\tau_1)$ at the instant $t = \tau_1$ from the condition

$$z^*(\tau_1) \in \{t = \tau_1\} \times (G^{(u)}(\{\tau_0, w^*, y^*, \mu^*, \nu^*\}, \tau_1, v^{(0)}[\cdot]) \cap N_{\tau_1})$$
$$v^{(0)}[t] = v^*(z(t_*), z^*(t_*), \delta)$$

Such a position can always be found because the system N_t is strongly u-stable (see (2.2)).

The process of obtaining $z(\tau)$ and $z^*(\tau)$ is repeated further, but now for $t_* = \tau_1$, etc., until the instant $t = \vartheta$. The first player's control

$$u_\Delta[t] = u_*(z(\tau_i), z^*(\tau_i), \delta), \quad \tau_i < t < \tau_{i+1}$$
$$i = 0, 1, \ldots, m(\Delta) - 1$$

thus constructed does not violate the constraint $J_1(u_\Delta[\cdot], t_*, \vartheta) \leqslant \mu_*$. The first player's control with a guide strategy constructed relative to system N_τ is denoted by

$$U(N_\tau) = U(\tau_i, \tau_{i+1}, r_1(\tau_i), r_2(\tau_i), N_\tau)$$

where

$$r_1(t_*) = \{w_*, y_*, \mu_*, \nu_*\}, \quad r_2(t_*) = \{w^*, y^*, \mu^*, \nu^*\}$$

Let $\{\alpha_j\}$ be a fixed sequence of numbers, possessing the following properties (see the analogous constructions in [5]):

$$0 < \alpha_j < 1 \tag{2.5}$$

$$\sum_{j=1}^{\infty} \alpha_j^2 \left\| \frac{\partial \omega_j}{\partial \nu_A} \right\|^2_{L_2(\Gamma)} < \infty, \quad \sum_{j=1}^{\infty} \alpha_j^2 \lambda_j^2 < \infty$$

Here ω_j, λ_j is a solution in space $H^1(\Omega)$ of the following spectral problem [10, 12, 13]:

$$A\omega = \lambda\omega, \quad \omega|_\Gamma = 0 \tag{2.6}$$

For simplicity we assume $a > 0$ (see (1.4)). By virtue of the constraints imposed on the operator A (see [10, 13]) and on domain Ω (see [10]) problem (2.6) has a solution from $H_0^1(\Omega) \cap H^2(\Omega)$ for a denumerable number of values of λ, and [10]

1) the λ_j are real, $0 < \lambda_1 < \ldots < \lambda_n < \ldots$, $\lambda_j \to \infty$ as $j \to \infty$;

2) $\int_\Omega \omega_j^2 \, dx = 1$ and $\int_\Omega \omega_j \omega_k \, dx = 0$ for $j \neq k$;

3) the ω_j form a basis in $L_2(\Omega)$ and in $H_0^1(\Omega)$.

Sequences (2.4) exist because by the theorems on traces [12, 13] the function $\partial\omega_j / \partial\nu_A \in L_2(\Gamma)$.

Note 2.1. In contrast to the conditions in [5] on the numbers $\{\alpha_j\}$, here we introduce an additional assumption (the last inequality in (2.5)). This is necessary because the constraints on the control resources now are integral.

By $\|\cdot\|_\alpha$ we denote a new norm in space $L_2(\Omega)$ defined as follows:

$$\|y\|_\alpha = \left(\sum_{j=1}^{\infty} \alpha_j^2 \langle y, \omega_j \rangle^2 \right)^{1/2}$$

In space H we introduce the norm

$$\|r\|_\alpha = \|\{w, y, \mu, \nu\}\|_\alpha = (\|w\|^2_{R_m} + \|y\|_\alpha^2 + \mu^2 + \nu^2)^{1/2}$$

By M_α^ε we denote the set $\{\varphi \in L_2(\Omega) \mid$, $m(\varphi) \in M$ can be found such that $\|\varphi - m(\varphi)\|_\alpha \leqslant \varepsilon\}$. The following lemma is valid.

Lemma 2.1. Let $N_t \neq \varnothing$, $t_0 \leqslant t \leqslant \vartheta$, $N_\vartheta \subset M^*$, and the system of sets N_t be strongly u-stable; then for any positive number $\varepsilon > 0$ we can find numbers $\delta(\varepsilon) > 0$ and $\beta(\varepsilon) > 0$ such that for all motions

$$y_t(x)_\Delta = y(x, t; t_0, w_0, y_0, \mu_0, \nu_0, U(N_\tau))_\Delta$$

there holds the inclusion $y_\vartheta(\cdot)_\Delta \in M_\alpha^\varepsilon$ if only $\delta(\Delta) \leqslant \delta(\varepsilon)$ and $\|r_1(t_0) - r_2(t_0)\|_\alpha \leqslant \beta(\varepsilon)$. Here

$$r_1(t_0) = \{w_0, y_0, \mu_0, \nu_0\}, \quad r_2(t_0) = \{w_0^*, \cdot y_0^*, \mu_0^*, \nu_0^*\}$$

is the guide's state at instant $t = t_0$.

The lemma's proof is based on the estimate of the distance between the motion of system (1.1), (1.3) and the motion of the guide; it can be established that the choice of the controls $u_*(z, z^*, \delta)$ and $v^*(z, z^*, \delta)$ ensures the proximity of these motions in the metric $\|\cdot\|_\alpha$ for a sufficiently small partitioning step δ. By $U^\circ(N_\tau)$ we denote the first player's strategy under which $r_1(t_0) = r_2(t_0)$. From Lemma 2.1 follows

Theorem 2.1. Let $N_t \neq \varnothing$, $t_0 \leqslant t \leqslant \vartheta$, $N_\vartheta \subset M^*$ and the system of sets N_t be strongly u-stable and let $r(t_0) = \{w_0, y_0, \mu_0, \nu_0\} \in N_{t_0}$; then strategy $U^\circ(N_\tau)$ solves the encounter problem.

3. Let us describe the procedure of the second player's position control with a guide for solving the evasion problem. Let a system of sets K_t, $t_0 \leqslant t \leqslant \vartheta$, be specified in space H Similarly as in Sect. 2 we introduce the sets $G^{(v)}(z_*, t^*, u[\cdot])$, where $z_* = \{t_*, w_*, y_*, \mu_*, \nu_*\}$, $t^* > t_*$, and we define the strong v-stability of system K_t. (It is only necessary to interchange u and v and μ and ν in the definition in Sect. 2.) By $u^*(z, z^*, \delta)$ we denote the function maximizing integral (2.3) for $J_1^{p_1}(u, t_*, t_* + \delta) \leqslant \mu^{*p_1} - \mu^{p_1}$, $\mu^* - \mu > 0$, $\|b\| \neq 0$ and $u^*(z, z^*, \delta) = 0$ when $\mu^* - \mu \leqslant 0$ or when $\|b\| = 0$. The function $v_*(z, z^*, \delta)$ minimizes integral (2.4) for $J_2^{p_2}(v, t_*, t_* + \delta) \leqslant \nu^{p_2} - \nu^{*p_2}$, $\nu - \nu^* > 0$, $\|c\| \neq 0$ and $v_*(z, z^*, \delta) = 0$ when $\nu - \nu^* \leqslant 0$ or when $\|c\| = 0$ The notation is the same as in Sect. 2. For the strongly v-stable family K_t we define the second player's control with a guide procedure. The second player's control is formed in the following manner:

$$v_\Delta[t] = v_*(z(\tau_i), z^*(\tau_i), \delta), \quad \tau_i \leqslant t < \tau_{i+1} = \tau_i + \delta \qquad (3.1)$$
$$i = 0, 1, \ldots, m(\Delta) - 1$$

Here $z(\tau_i)$ is the game's position realized at instant $t = \tau_i$ under the choice of control $v_\Delta[t]$ from (3.1) in pair with an admissible control $u(t)$ $(\tau_0 \leqslant t < \tau_i)$ of the first player, i.e. $J_1(u, \tau_0, \tau_i) \leqslant \mu(\tau_0)$; $z^*(\tau_i)$ is the guide's position at instant $t = \tau_i$. To determine the guide's position we use the controls

$$u^{[i]}(t) = u^*(z(\tau_i), z^*(\tau_i), \delta), \tau_i \leqslant t < \tau_{i+1}, \ i = 0, 1, \ldots, m(\Delta) - 1$$

As the initial position we select an arbitrary point of set K_{t_0}. The succeeding positions of the guide are determined from the condition

$$z^*(\tau_i) \in \{t = \tau_i\} \times (G^{(v)}(z^*(\tau_{i-1}), \tau_i, u^{[i-1]}(\cdot)) \cap K_{\tau_i})$$

up to the instant $t = \vartheta$. Such points $z^*(\tau_i)$ always exist because system K_t is strongly v-stable. The control $v_\Delta[t]$ constructed does not violate the constraints $J_2(v_\Delta[t], t_0, \vartheta) \leqslant \nu(t_0)$. The strategy $V^\circ(K_\tau)$ is defined as in Sect. 2.

In what follows we shall examine only those systems of strongly v-stable sets K_t, $t_0 \leqslant t \leqslant \vartheta$, for which $K_\vartheta \subset G^*$, where $G^* = \{\{w, y, \mu, \nu\} \in H | \mu \geqslant 0, \nu \geqslant 0, y \in G, G = G \subset L_2(\Omega), G \cap M = \varnothing\}$. There holds the following

Theorem 3.1. Let a strongly v-stable system K_t, $t_0 \leqslant t \leqslant \vartheta$ exist such that $\{w_0, y_0, \mu_0, \nu_0\} \in K_{t_0}$; then strategy $V^\circ(K_\tau)$ solves the evasion problem from the position $\{t_0, w_0, y_0, \mu_0, \nu_0\}$.

The proof is similar to that of Lemma 2.1 and of Theorem 2.1 for strongly v-stable sets. The following statements concerning the solution of the evasion problem are valid.

Lemma 3.1. If the position $z_* = \{t_*, w_*, y_*, \mu_*, \nu_*\}$ belongs to some strongly v-stable family K_t, $t_* \leqslant t \leqslant \vartheta$, then an ε-neighborhood of this position in space

H exists such that the evasion problem is solvable from any position in this ε-neighborhood.

The proof follows from the analogy of Lemma 2.1 for strongly v-stable sets. The validity of the next statement can be proved by using Lemma 3.1.

Lemma 3.2. If the position $z_* = \{t_*, w_*, y_*, \mu_*, \nu_*\}$ belongs to some strongly v-stable family K_t, $t_* \leqslant t \leqslant \vartheta$, then a strongly v-stable family $K_t{}^*$ exists such that $K_t{}^*$ wholly contains some ε-neighborhood of point z_*.

Let us consider the following family of sets: $K_t{}^{(v)} = \bigcup K_t$ is the union of all strongly v-stable families. We denote $N_t = H \setminus K_t{}^{(v)}$. There holds

Theorem 3.2. Let $N_{t_0} \neq \varnothing$; then $N_t \neq \varnothing$, $t_0 \leqslant t \leqslant \vartheta$, and the system of sets N_t is strongly u-stable.

The following theorem on the alternative implies from Theorems 2.1 and 3.2 and Lemmas 3.1 and 3.2.

Theorem 3.3. Either the encounter problem or the evasion problem is always solvable for any initial position $\{t_0, w_0, y_0, \mu_0, \nu_0\}$. The encounter (evasion) problem is solvable if and only if

$$\{w_0, y_0, \mu_0, \nu_0\} \in N_{t_0} \quad (\{w_0, y_0, \mu_0, \nu_0\} \notin N_{t_0})$$

Note 3.1. All the constructions considered above extend to the second and third boundary-value problems [11—13] for Eq. (1.1). Similar constructions are implementable for the case of instantaneous constraints on the player's controls. The last of conditions (2.5) may be absent. Similar results are valid for systems with distributed controls of the form

$$\partial y / \partial t + A y = f + b u_1 + c v_1$$

where the constraints on controls u_1 and v_1 are of type (1.5). Finally, we note that the results presented above hold for the problems of encounter and evasion by the instsnt ϑ (see [2]).

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